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14. ABSTRACT

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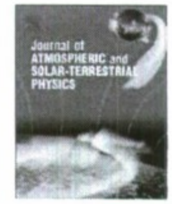

Estimates of lifetimes against pitch angle diffusion

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We consider timescales on which particle distributions respond to pitch angle diffusion. On the longest timescale, the distribution decays at a single rate independent of equatorial pitch angle α_0, even though the diffusion coefficient, and the distribution itself, may vary greatly with α_0. We derive a simple integral expression to approximate this decay rate and show that it gives good agreement with the full expression. The roles of both the minimum and loss cone values of the diffusion coefficient are demonstrated and clarified.

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1. Introduction

A dominant process for radiation belt electrons is pitch angle diffusion, caused by cyclotron and Landau resonance with a variety of plasma waves. The resulting scattering into the loss cone and precipitation into the atmosphere is a major loss mechanism, especially inside of geosynchronous orbit. Neglecting all other processes, the relaxation of a population injected with an arbitrary pitch angle distribution can be considered to occur on a sequence of time scales given by the ordered eigenvalues of the bounce-averaged pitch angle diffusion operator (e.g., Schulz and Lanzerotti, 1974; see also, O'Brien et al., 2008). Although the long time behavior is described by the single, longest timescale, the time required for the higher eigenmodes to decay can be significant, and can lead to complications in interpreting short sequences of observations (Kress et al., 2007; Baker et al., 2007). In this paper, we generally refer to the slowest diffusion time scale as "the lifetime."

Strictly, this eigenvalue problem depends on the values of the diffusion coefficients at all values of equatorial pitch angle α_0. These coefficients can vary widely with α_0, as different resonances come into play depending on the local values of particle pitch angle, wave refractive index, and geomagnetic field. Typically, for a given wave mode, diffusion by cyclotron resonances is the largest for small α_0 while diffusion by the Landau resonance is the largest for large α_0, leading to a pronounced minimum at some intermediate α_0 (although this may be masked by combining different wave modes with diffusion minima located at different α_0 values). Lyons and Thorne (1973) gave an integral equation for simultaneously determining the lowest eigenvalue and eigenfunction, or the lifetime and relaxed pitch angle distribution, for an arbitrary profile of $D(\alpha_0)$. Albert (1994) reformulated this as a boundary value problem for a set of ordinary differential equations, readily solved with "shooting" method (Press et al., 1992). Both approaches will be used below.

Shprits et al. (2006) proposed a simple, convenient method of estimating the global diffusion lifetime, namely as the inverse of the diffusion coefficient evaluated at the edge of the loss cone (denoted by D_L). This follows in the footsteps of several earlier papers which related the lowest eigenvalue to D_L for simple model problems (e.g., Kennel and Petschek, 1966; Roberts, 1969; Schulz and Lanzerotti, 1974). Shprits et al. (2006) argued that away from the loss cone, small diffusion coefficients are compensated by the development of large gradients in the particle distribution. A series of numerical experiments with model diffusion coefficients showed reasonable

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agreement for $D(\alpha_0)$ profiles with moderate internal minima. They also noted that large minima lead to slow relaxation of the pitch angle profile, perhaps slower than of physical interest, while the behavior on shorter time scales did seem to be controlled by the loss cone diffusion rate. They concluded that internal minima in D were subordinate to the loss cone value as long as these diffusion coefficients were within a factor of 10.

On the other hand, one might expect the decay rate to be dictated by the minimum value of the diffusion rate, termed a "bottleneck" by Schulz (1991) who tied them to "shoulders" in the distribution function $f(\alpha_0)$. Lichtenberg and Lieberman (1983), discussing transport with variable diffusion coefficients in a different context, also noted that the gradients becomes steeper where the diffusion is slower but concluded from the continuity of flow that the effective diffusion rate is given by the harmonic average, $\langle (1/D)\rangle^{-1}$, and that this is dominated by regions where D is the smallest. This seems consistent with the numerical results of Abel and Thorne (1998), who found that diffusion due to lightning whistlers and VLF transmitters, while negligible compared to Coulomb collisions and hiss rates at the loss cone, greatly reduce the lifetime of 500 keV electrons at $L = 2.4$ by smoothing the deep minimum that would otherwise occur at $\alpha_0 \approx 65^\circ$. Similarly a numerical study of diffusion due to combined whistler mode hiss and EMIC waves (Albert, 2003) found the electron lifetimes to be strongly affected by the hiss amplitudes which determined the value (and location) of the minimum in D, even though hiss was negligible at theloss cone itself.

To explore and reconcile these two viewpoints, we start with exact expressions for the decay rates and derive an integral approximation which reflects aspects of both. We also evaluate a model that admits a (nearly) exact analytical treatment. Section 3 tests the approximation and several simpler expressions against an analytically solvable model, and Section 4 does the same with a series of numerical experiments paralleling those of Shprits et al. (2006). We conclude that the integral approximation is an accurate and reliable estimate for a wide variety of diffusion models.

2. General expressions for lifetimes

We start with the one-dimensional bounce-averaged pitch angle diffusion equation,

$$\frac{\partial f}{\partial t} = \frac{1}{T\sin 2\alpha_0}\frac{\partial}{\partial \alpha_0}\left(DT\sin 2\alpha_0 \frac{\partial f}{\partial \alpha_0}\right) + S, \tag{1}$$

where α_0 is the equatorial pitch angle, D is the corresponding pitch angle diffusion coefficient, S is a stationary source term, and $T \approx 1.30 - 0.56 \sin\alpha_0$ is the normalized bounce time. The boundary conditions are $f = 0$ at the loss cone angle α_L and $\partial f/\partial\alpha_0 = 0$ at $\alpha_0 = \pi/2$.

Following Lyons and Thorne (1973), we write $f(\alpha_0, t) = f_0(\alpha_0) + g(\alpha_0)\exp(-t/\tau)$, where f_0 solves Eq. (1) in steady state, to get

$$\frac{-gT\sin 2\alpha_0}{\tau} = \frac{d}{d\alpha_0}\left(DT\sin 2\alpha_0\frac{dg}{d\alpha_0}\right). \tag{2}$$

Integrating from the loss cone angle α_L to $\pi/2$ and taking the boundary conditions into account gives

$$\frac{1}{\tau}\int_{\alpha_L}^{\pi/2} gT\sin 2\alpha'\,d\alpha' = D_L T_L \sin 2\alpha_L \left.\frac{dg}{d\alpha_0}\right|_{\alpha_L}, \tag{3}$$

where $D_L = D(\alpha_L)$ and $T_L = T(\alpha_L)$. Eq. (3) is the same as Eq. (13) of Lyons and Thorne. More generally, integrating from an arbitrary value of α to $\pi/2$ gives

$$\frac{1}{\tau}\int_{\alpha}^{\pi/2} gT\sin 2\alpha'\,d\alpha' = DT\sin 2\alpha\frac{dg}{d\alpha}. \tag{4}$$

Dividing Eq. (3) by Eq. (4), solving for $dg/d\alpha$, and integrating from α_L to α_0 gives

$$g(\alpha_0) = \int_{\alpha_L}^{\alpha_0} d\alpha \frac{D_L T_L \sin 2\alpha_L (dg/d\alpha)|_{\alpha_L}}{DT\sin 2\alpha}\Lambda(\alpha), \tag{5}$$

where

$$\Lambda(\alpha) = 1 - \frac{\int_{\alpha_L}^{\alpha} gT\sin 2\alpha'\,d\alpha'}{\int_{\alpha_L}^{\pi/2} gT\sin 2\alpha'\,d\alpha'}. \tag{6}$$

Eq. (5) is the same as Eq. (12) of Lyons and Thorne (1973). Combining this with Eq. (3) gives

$$\tau = \int_{\alpha_L}^{\pi/2} d\alpha_0 T\sin 2\alpha_0 \int_{\alpha_L}^{\alpha_0}\frac{d\alpha}{DT\sin 2\alpha}\Lambda(\alpha). \tag{7}$$

We denote by τ_0 the value of τ obtained with the approximation $T(\alpha_0) = 1$. Making this approximation in Eq. (7) and reversing the order of integration according to $\int_{\alpha_L}^{\pi/2} d\alpha_0 \int_{\alpha_L}^{\alpha_0} d\alpha = \int_{\alpha_L}^{\pi/2} d\alpha \int_{\alpha}^{\pi/2} d\alpha_0$ yields

$$\tau \approx \tau_0 = \int_{\alpha_L}^{\pi/2} d\alpha \frac{\cos\alpha}{2D\sin\alpha}\Lambda(\alpha). \tag{8}$$

3. An estimate

We now consider the situation where D has a deep minimum at some value $\alpha = \alpha_*$. As α decreases from $\pi/2$, the left-hand side of Eq. (4) increases. If D becomes small at $\alpha = \alpha_*$, then $dg/d\alpha$ must become large there, so that g is much larger for $\alpha > \alpha_*$ than for $\alpha < \alpha_*$. If $g(\alpha)\sin 2\alpha$ is approximated as 0 for $\alpha < \alpha_*$ and constant for $\alpha > \alpha_*$, Eq. (6) gives

$$\Lambda(\alpha) \approx \begin{cases} 1, & \alpha \leqslant \alpha_*, \\ 1 - (\alpha - \alpha_*)\Big/\left(\dfrac{\pi}{2} - \alpha_*\right), & \alpha > \alpha_* \end{cases} \tag{9}$$

so that

$$\tau \approx \int_{\alpha_L}^{\pi/2} d\alpha \frac{\cos\alpha}{2D\sin\alpha} - \int_{\alpha_*}^{\pi/2} d\alpha \frac{\cos\alpha}{2D\sin\alpha}\frac{\alpha - \alpha_*}{(\pi/2) - \alpha_*}. \tag{10}$$

Because the second integrand is zero at α_*, where $1/(D\sin\alpha)$ is assumed to have a steep maximum, the second integral will be less than the first integral, finally yielding the estimate

$$\tau_* \equiv \int_{\alpha_L}^{\pi/2} d\alpha \frac{\cos\alpha}{2D\sin\alpha}. \tag{11}$$

This formula clearly suggests that τ will be determined mostly by the minimum value of $2D\tan\alpha_0$.

This partially vindicates both simple estimates discussed above. If the minimum of D is deep enough, it will largely determine the minimum of $D\tan\alpha_0$. If $D(\alpha_0)$ is relatively flat, the minimum of $D\tan\alpha_0$ will occur at the loss cone. If the minimum of D occurs near the loss cone, the two concepts coincide.

4. An analytically solvable model

For some simple analytical forms of $D(\alpha_0)$, and neglecting the variation of $T(\alpha_0)$, it is possible to write the complete set of eigenvalues and eigenfunctions of the diffusion operator in closed form. As given by Schulz and Lanzerotti (1974), with slight change of notation, the model $D = D_0\cos^{2\sigma}\alpha_0/\sin^2\alpha_0$ with $\sigma<1$ yields (as we have verified) the eigenvalues and (unnormalized) eigenfunctions

$$\lambda_n = \frac{(1-\sigma)^2\kappa_{\nu n}^2}{\cos^{2-2\sigma}\alpha_L}D_0, \quad n=0,1,2,\ldots$$
$$g_n(\alpha_0) = \frac{\cos^\sigma\alpha_L}{\cos^\sigma\alpha}J_\nu\left[\kappa_{\nu n}\frac{\cos^{1-\sigma}\alpha}{\cos^{1-\sigma}\alpha_L}\right], \tag{12}$$

where $\nu = \sigma/(1-\sigma)$ and $\kappa_{\nu n}$ is the nth zero of the Bessel function J_ν. Remarkably, the expressions remain valid for $\sigma<0$, in which case D has an internal minimum at $\sin^2\alpha_0 = 1/(1-\sigma)$; the minimum value is $D_0(1-\sigma)(-\sigma/(1-\sigma))^\sigma$. Also, $\sigma<0$ guarantees that the eigenfunctions obey the boundary conditions associated with Eq. (1).

With $\sigma=-1$, $J_\nu(z) = (2/\pi z)^{1/2}\cos z$ (Gradshteyn and Ryzhik, 1980), and the eigenvalues and eigenfunctions become

$$\lambda_n = \frac{\pi^2(1+2n)^2}{\cos^4\alpha_L}D_0, \quad n=0,1,2,\ldots$$
$$g_n(\alpha_0) = \cos\left[\frac{\pi(1+2n)}{2}\frac{\cos^2\alpha_0}{\cos^2\alpha_L}\right]. \tag{13}$$

These results agree closely with full numerical solutions based on Albert (1994), and essentially perfectly with numerical solutions setting $T(\alpha_0)=1$.

The inverse of the lowest eigenvalue is

$$\tau_0 = \frac{\cos^4\alpha_L}{\pi^2 D_0}. \tag{14}$$

From Eq. (11), the estimate τ_* is

$$\tau_* = \frac{\cos^4\alpha_L}{8D_0}, \tag{15}$$

which agrees extremely well with τ_0, even though the minimum of this model $D(\alpha_0)$ is not sharp enough to cause a pronounced "shoulder" in $g(\alpha_0)$ as assumed. The inverse minimum values

$$\frac{1}{D_{\min}} = \frac{1}{4D_0} \tag{16}$$

and

$$\frac{1}{(2D\tan\alpha_0)_{\min}} = \frac{3\sqrt{3}}{32D_0} \approx \frac{1}{6D_0} \tag{17}$$

also usually agree with τ_0 to within about a factor of 2 for plausible values of α_L. The inverse value at the edge of the loss cone,

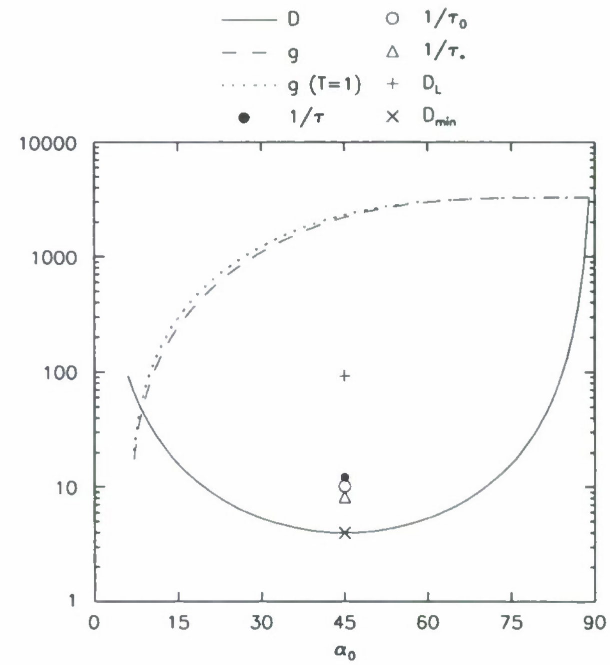

Fig. 1. Diffusion coefficient D and pitch angle profile g vs. equatorial pitch angle α_0 for the analytical model discussed in the text. Also shown are the decay lifetime τ and several estimates.

$$\frac{1}{D_L} = \frac{\cos^2\alpha_L\sin^2\alpha_L}{D_0}, \tag{18}$$

differs by the factor ~$10\tan^2\alpha_L$, which can be significantly less than 1.

Fig. 1 shows $D(\alpha_0)$ and the shape of $g(\alpha_0)$, calculated with α_L corresponding to $L=4$ in a dipole magnetic field. The approximation $T(\alpha_0)=1$ has virtually no effect on the shape of g. Also shown are the value of τ from a full computation, the analytic value τ_0, and the estimate τ_*, all of which agree quite closely, as well as the simpler estimates D_L and D_{min}. Both simpler estimates are seen to give significantly poorer agreement than τ_*.

5. Numerical experiments

To further test the estimate given by Eq. (11), we repeat the numerical experiments of Shprits et al. (2006). These were based on pitch angle diffusion coefficients for 1 MeV electrons at $L=4$ due to lower-band chorus waves, calculated using the parallel-propagation formulation of Summers (2005). (The parameters used were $\omega_M = 0.35\Omega_e$, $\delta\omega = 0.15\Omega_e$, $\omega_{LC} = 0.05\Omega_e$, $\omega_{UC} = 0.65\Omega_e$, and $\omega_{pe}/\Omega_e = 4.15$.) The D values were combined with artificial step-function decreases at various α_0 ranges. For each modified version of D, we computed the lowest eigenmode $g(\alpha_0)$ and lifetime τ according to the shooting code algorithm of Albert (1994), and repeated the calculation with the approximation $T(\alpha_0)=1$ to obtain τ_0. The lifetimes were compared to the approximation τ_* and the simple estimates D_L^{-1} and $D_{\min}^{-1}$.

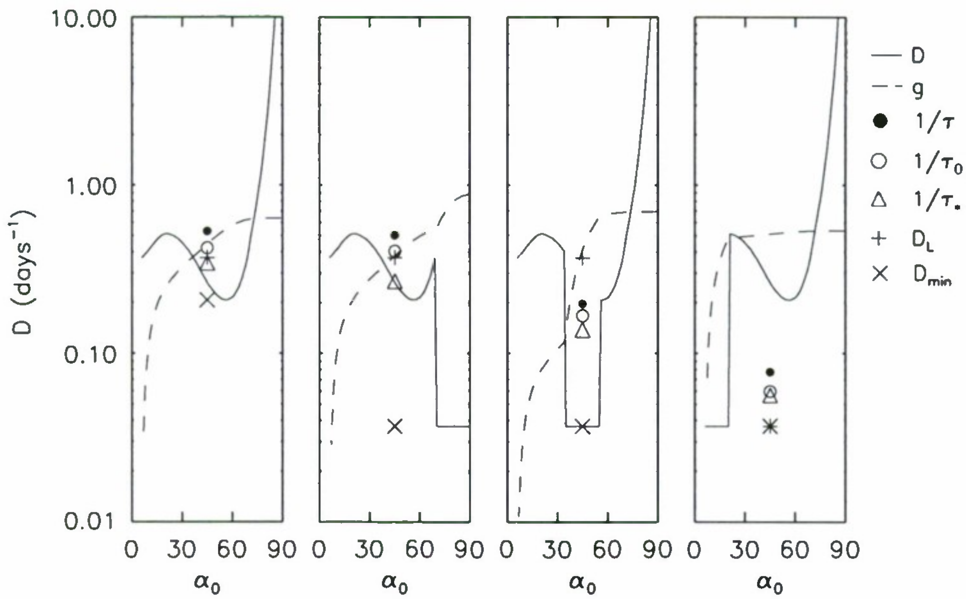

Fig. 2. Diffusion coefficients based on parallel-propagating chorus waves combined with step decreases at various locations. Also shown are the lifetime and various estimates; the symbols have the same meaning as in Fig. 1.

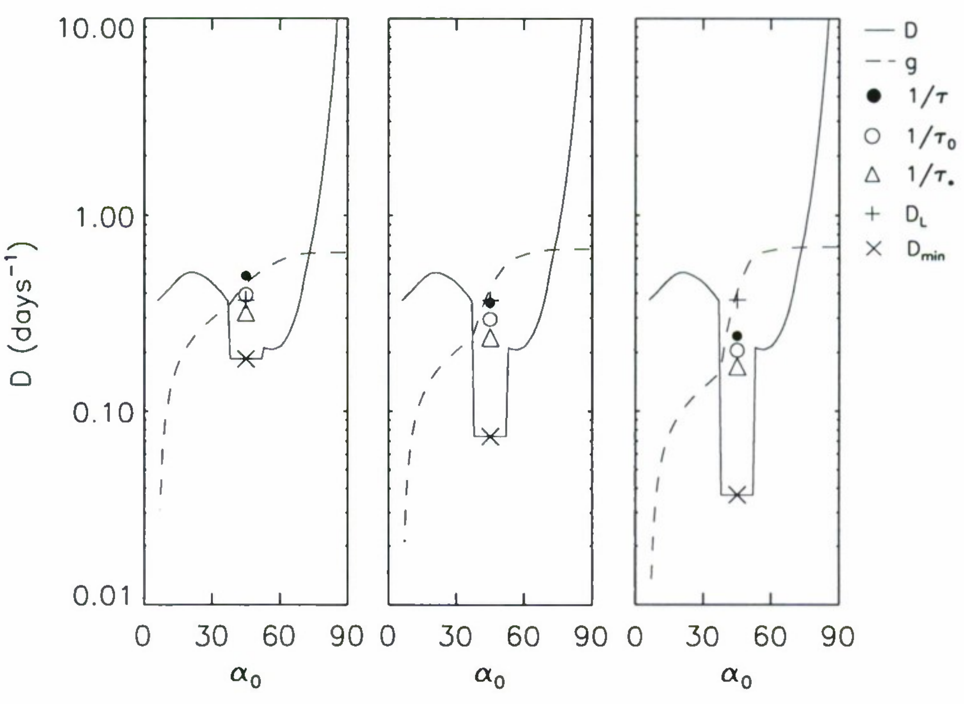

Fig. 3. Same as Fig. 2 but with step decreases of various depths.

Fig. 2 corresponds to Fig. 3 of Shprits et al. (2006), where a step of depth $D_L/10$ and width 20° was imposed at large, intermediate, and small values of α_0. In all cases, both τ and $g(\alpha_0)$ agree with the results found by Shprits et al. by directly evolving the diffusion equation in time numerically. The estimate D_L^{-1} works well if the step is at large or small α_0, or absent, but starts to break down when the step is at intermediate α_0. The estimate $D_{\rm min}^{-1}$ is generally poor, but τ_* is consistently good. Actually, τ_* is a better estimate of τ_0 than of τ, since both assume $T(\alpha_0) = 1$, but the difference is small.

Fig. 3, corresponding to Fig. 4 of Shprits et al. (2006), shows a similar comparison with the step decrease held at intermediate α_0 but with varying depth. The value D_L^{-1} is a good approximation of τ for small and moderate step

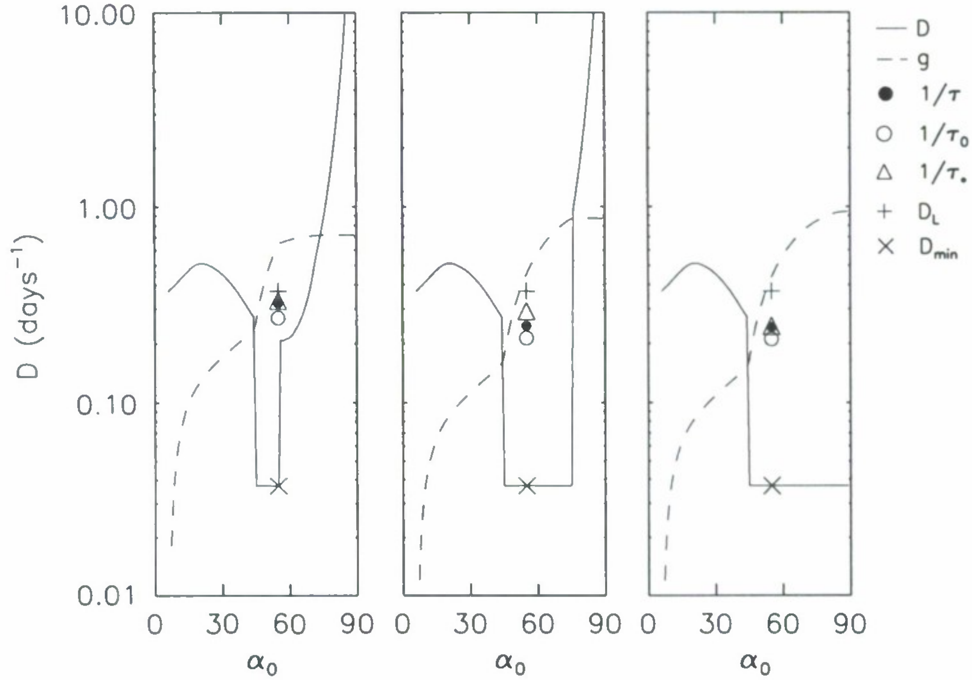

Fig. 4. Same as Fig. 2 but with step decreases of various widths.

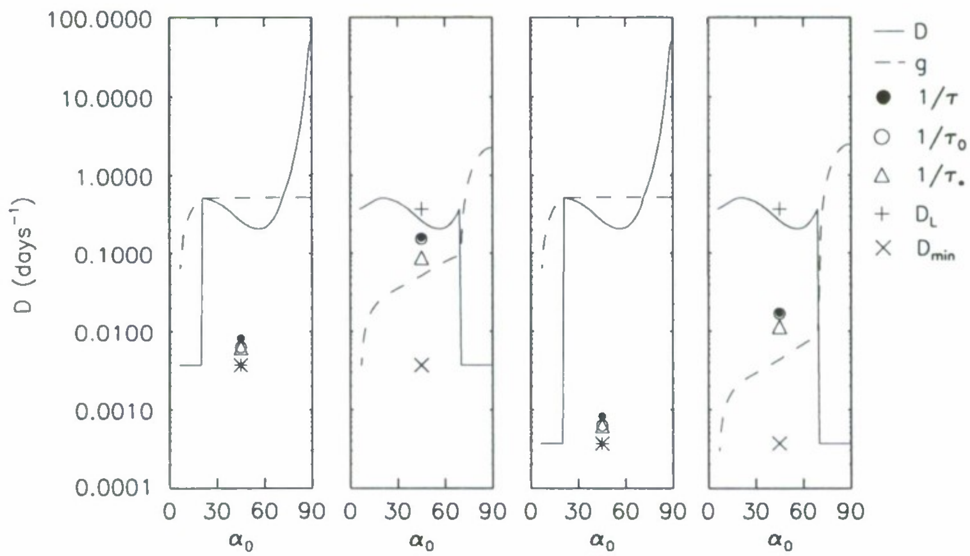

Fig. 5. Same as Fig. 2 but with large step decreases at large and small α_0.

decreases, but breaks down for large steps. D_{min}^{-1} is a poor estimate of τ, but τ_* is reliably good.

In Fig. 4, the width of the step is varied. Once again, τ and τ_0 are approximated poorly by D_{min}^{-1} and well by D_L^{-1} for narrow and moderate steps, but the agreement breaks down for wide steps, while τ_* provides a uniformly good estimate.

Finally, Fig. 5 shows much deeper steps ($D_L/100$ and $D_L/1000$). When the step is at small α_0, all the values are close together. However, when the step is at large α_0, only τ_* is a good approximation to τ and τ_0.

6. Conclusions

We have derived a new estimate of the lowest pitch angle diffusion timescale τ, namely τ_* of Eq. (11), based on the integral of $1/D\tan\alpha$, and verified its accuracy with an analytically solvable model and with a variety of numerical experiments. The approximation is simpler and more physically transparent than the full calculation, and allows convenient estimates of changing various wave parameters (such as amplitude or latitudinal extent) by considering D at only a certain values of α_0. The exact expression τ given by Eq. (7), however, is not difficult to compute (as in Albert, 1994), and is the most rigorous.

Based on their numerical experiments, Shprits et al. (2006) concluded that the estimate of τ by D_L^{-1} was often good but broke down for step function minima in D less than $D_L/10$. This makes sense in light of Eq. (11), since the integral will be dominated by the minimum value of $D\tan\alpha$, and that minimum will only be driven by a minimum in D itself where (for moderate α), $D < D_L \tan\alpha_L$. Since typically $\tan\alpha_L \approx 1/10$, a minimum in D would have to be less than $0.1D_L$ to be dominant. A similar conclusion can be drawn from the analytical results of Section 4: there, τ becomes substantially larger than D_L^{-1} when $\pi\sin\alpha_L < 1$, which implies $D_{\min} < (4/\pi^2)D_L \approx 0.4D_L$. As noted by Shprits et al. (2007), actual diffusion coefficients can exhibit deep minima, such as when EMIC waves are present in combination with whistler mode chorus waves.

It is also true that the resulting lowest order lifetimes may be longer than the duration of the waves, so that the particle population may not have time to settle into the lowest eigenmode. In these situations, and especially considering energy diffusion and radial transport, no single timescale may be sufficient to describe the dynamics in a meaningful way. Fortunately, the development of 2D and 3D codes to advance the time-dependent diffusion equation is well under way.

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